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Instantons in supersymmetric Yang–Mills and D-instantons in IIB superstring theory

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Abstract: The one-instanton contributions to various correlation functions of superconformal currents in four-dimensional $\mathcal{N} = 4$ supersymmetric $SU(2)$ Yang–Mills theory are evaluated to the lowest order in perturbation theory. Expressions of the same form are obtained from the leading effects of a single D-instanton extracted from the IIB superstring effective action around the $AdS_5 \times S^5$ background. This is in line with the suggested AdS/Yang–Mills correspondence. The relation between Yang–Mills instantons and D-instantons is further confirmed by the explicit form of the classical D-instanton solution in the $AdS_5 \times S^5$ background and its associated supermultiplet of zero modes. Speculations are made concerning instanton effects in the large-N_c limit of the $SU(N_c)$ Yang–Mills theory.

Keywords: Nonperturbative Effects, D-branes, Brane Dynamics in Gauge Theories, Extended Supersymmetry.

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Contents

1. Introduction

Four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills theory [1] is a very special quantum field theory. It is the original example of a theory possessing exact electromagnetic duality [2, 3, 4] which is connected to the fact that it contains an infinite set of stable dyonic BPS states [5] and also has a vanishing renormalization group β function [6]. Whereas the abelian theory is free, the nature of the non-abelian theory depends on whether scalar fields have vacuum expectation values. In the Coulomb phase, reached by giving vacuum expectation values (vev's) to the scalars in the Cartan subalgebra, the two-derivative Wilsonian effective action is believed not to be renormalized either in perturbation theory or by non-perturbative effects. The superconformal invariance of the theory is, however, broken in this phase. The phase in which all scalar fields have vanishing vev's is expected to describe a highly nontrivial superconformal field theory. Although it is very difficult to understand the nature of this phase from direct perturbative calculations, according to the recent flurry of work [7, 8, 9, 10] certain properties of the theory should be understandable in terms of type IIB superstring theory compactified on $AdS_5 \times S^5$, where the Yang–Mills theory is located on the four-dimensional boundary of the five-dimensional anti de-Sitter space.

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According to the proposal made by Maldacena in [8] properties of $SU(N_c)$ $\mathcal{N}=4$ Yang–Mills theory in the large-N_c limit may be determined by semi-classical approximations to the superstring theory. In this limit the boundary Yang–Mills theory is interpreted as the world-volume theory for a large-N_c collection of coincident D3-branes. The $AdS_5 \times S^5$ geometry is the near-horizon description of the classical D3-brane solution [11] which is a source of non-vanishing self-dual Ramond–Ramond ($R \otimes R$) five-form field strength, $F_5 = *F_5$ [12]. In fact, the D3-brane solution plays the rôle of an interpolating soliton between two maximally supersymmetric configurations (with $16+16$ supercharges) — flat ten-dimensional Minkowski space at infinity and $AdS_5 \times S^5$ at the horizon. The extra 16 Killing spinors at the horizon are in one-to-one correspondence with the special supersymmetry transformations of the boundary theory [13]. The $SO(6)$ isometry of the S^5 factor corresponds to the gauging of the $SU(4)$ R-symmetry group while the $SO(4,2)$ isometry of AdS_5 coincides with the conformal group of the boundary theory at the horizon.

In this picture the boundary values of the fields of the bulk superstring theory compactified on $AdS_5 \times S^5$ are sources that couple to gauge-invariant operators of the four-dimensional boundary $\mathcal{N}=4$ supersymmetric Yang–Mills theory. The lowest Kaluza–Klein modes of the graviton supermultiplet couple to the superconformal multiplet of Yang–Mills currents. These fields and currents will be reviewed in more detail in section 2. More generally, all the Kaluza-Klein excitations of the bulk supergravity theory can be put in one-to-one correspondence with gauge-singlet composite Yang–Mills operators [9, 10, 14, 15].

According to this idea the effective action of type IIB supergravity, evaluated on a solution of the equations of motion with prescribed boundary conditions, is equated with the generating functional of connected gauge-invariant correlation functions in the Yang–Mills theory. The parameters of the $\mathcal{N}=4$ Yang–Mills theory and the IIB superstring on $AdS_5 \times S^5$ are related by

$$g_s = \frac{g^2_{_{YM}}}{4\pi}\,, \qquad 2\pi\tilde{C}^{(0)} = \theta_{_{YM}}\,, \qquad \frac{L^2}{\alpha'} = \sqrt{g^2_{_{YM}} N_c}\,, \tag{1.1}$$

where $g_s = e^{\tilde{\phi}}$ is the string coupling ($\tilde{\phi}$ is the constant dilaton), $\tilde{C}^{(0)}$ is the constant $R \otimes R$ axionic background, $g_{_{YM}}$ is the Yang–Mills coupling, $\theta_{_{YM}}$ is the vacuum angle and L is the radius of both the AdS_5 and S^5 factors of the bulk background. The complex Yang–Mills coupling is therefore identified with the constant boundary value of the complex scalar field of the IIB superstring,

$$\frac{\theta_{_{YM}}}{2\pi} + \frac{4\pi i}{g^2_{_{YM}}} = \tilde{C}^{(0)} + \frac{i}{g_s}\,. \tag{1.2}$$

Most of the tests of this conjecture have so far amounted to the computation of two-point and three-point correlations of currents [16] based on the semiclassical approximation to the bulk supergravity ($g_s << 1$) which is valid at length scales much larger than

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the string scale or, equivalently, in the limit $\alpha'/L^2 << 1$. This is the limit $N_c \to \infty$ and $g_{_{YM}} \to 0$ with $g^2_{_{YM}} N_c >> 1$. This can also be viewed as the large-N_c limit introduced by 't Hooft [17] with the coupling $\hat{g}^2 = g^2_{_{YM}} N_c$ fixed at a large value.

The explicit connection between the bulk theory and the boundary theory can be expressed symbolically as [10, 9, 18]

$$\exp(-S_{_{IIB}}[\Phi_m(J)]) = \int DA \, \exp(-S_{_{YM}}[A] + \mathcal{O}_\Delta[A]J) \,, \tag{1.3}$$

where $S_{_{IIB}}$ is the effective action of the IIB superstring or its low energy supergravity limit which is evaluated in terms of the 'massless' supergravity fields and their Kaluza–Klein descendents, that we have generically indicated with $\Phi(z;\omega)$, where ω are the coordinates on S^5 and $z^M \equiv (x^\mu, \rho)$ ($M = 0,1,2,3,5$ and $\mu = 0,1,2,3$) are the AdS_5 coordinates ($\rho \equiv z_5$ is the coordinate transverse to the boundary). The notation in (1.3) indicates that the action depends on the boundary values, $J(x)$, of the bulk fields. The fluctuating boundary $\mathcal{N} = 4$ supersymmetric Yang–Mills fields are denoted by A and $\mathcal{O}(A)$ in (1.3) is the set of gauge-invariant composite operators to which J couples. The recipe for computing correlations involves the 'bulk-to-boundary' Green functions which are defined as specific normalized limits of bulk-to-bulk Green functions [9, 10, 19] when one point is taken to the AdS boundary. The precise forms of these propagators depend on the spin and mass of the field. For example, the normalized bulk-to-boundary Green function for a dimension Δ scalar field is given by

$$G_\Delta(x, \rho, \omega; x', 0, \omega') = c_{_\Delta} K_\Delta(x^\mu, \rho; x'^\mu, 0) \,, \tag{1.4}$$

which is independent of ω and where $c_{_\Delta} = \Gamma(\Delta)/(\pi^2\Gamma(\Delta-2))$ and

$$K_\Delta(x^\mu, \rho; x'^\mu, 0) = \frac{\rho^\Delta}{(\rho^2 + (x-x')^2)^\Delta} \,. \tag{1.5}$$

The expression (1.4) is appropriate for an 'S-wave' process in which there are no excitations in the directions of the five-sphere, S^5. In terms of K_Δ the bulk field

$$\Phi_m(z; J) = c_{_\Delta} \int d^4x' K_\Delta(x, \rho; x', 0) J_\Delta(x') \tag{1.6}$$

satisfies the boundary condition as $\rho \to 0$,

$$\Phi_m(x, \rho; J) \approx \rho^{4-\Delta} J_\Delta(x) \,, \tag{1.7}$$

since $\rho^{\Delta-4} K_\Delta$ reduces to a δ-function on the boundary. The conformal dimension of the operator is related to the AdS mass of the corresponding bulk field by $(mL)^2 = \Delta(\Delta-4)$, so that $\Delta_\pm = 2 \pm \sqrt{4 + (mL)^2}$ and only the positive branch, $\Delta = \Delta_+$, is relevant for the lowest-'mass' supergravity multiplet. In the case of a massless scalar field ($\Delta_+ = 4$) the propagator reduces to $\delta^{(4)}(x^\mu - x'^\mu)$ in the limit $\rho \to 0$.

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For our considerations it will prove crucial in the following that the expression (1.5) in the case $\Delta_+ = 4$ has exactly the same form as the contribution of a Yang–Mills instanton to $\mathrm{Tr}(F^-_{\mu\nu})^2$ (where $F^-_{\mu\nu}$ is the non-abelian self-dual field strength) when the fifth coordinate ρ is identified with the instanton scale. At the same time, we will see that in this case (1.5) has precisely the same form as the five-dimensional profile of a D-instanton centered on the point z^M in AdS_5 and evaluated at the boundary point $(x'^\mu, 0)$. This is a key observation in identifying D-instanton effects of the bulk theory with Yang–Mills instanton effects in the boundary theory. It is related to the fact that the moduli space of a Yang–Mills instanton has an AdS_5 factor.

Two-point and three-point correlation functions of superconformal currents are not renormalized from their free-field values due to the $\mathcal{N} = 4$ superconformal invariance so they do not get interesting interaction corrections [19, 20, 21, 22]. However, higher-point correlation functions do receive nontrivial interaction corrections. Here we will be concerned with calculations of processes in which the one-instanton contributions can be evaluated exactly to lowest order in perturbation theory. The literature on this subject includes examples in which all scalar field vacuum expectation values are zero ($v = 0$) [23] as well as those in which the scalar fields in the Cartan subalgebra have non-vanishing vacuum expectation values ($v \neq 0$) [24]. In the latter case the theory is in an abelian Coulomb phase where the dynamics is rather trivial. The superconformal theory of interest here has $v = 0$ and cannot be obtained simply as the limit of the theory with $v \neq 0$, since all the higher derivative terms that naturally have inverse powers of v become singular. Although there are few explicit computations of nontrivial instanton effects in the literature in the superconformal $\mathcal{N} = 4$ case where $v = 0$ (see, however [25, 26]), we will see that there is no impediment to using well known methods [23] to perform such calculations.

In section 3 we will evaluate the contributions due to a single $SU(2)$ instanton to specific correlation functions in the Yang–Mills theory. One class of correlation functions that we will discuss is the product of four bilinear operators in the current supermultiplet, $\langle \mathcal{W}_{(2)}(x_1) \dots \mathcal{W}_{(2)}(x_4) \rangle$. For example, we will consider the correlator of the components, $\mathcal{Q}$, of $\mathcal{W}_{(2)}$ that are bilinears in the scalar fields φ belonging to the six-dimensional representation of the $SU(4)$ R-symmetry. This correlation function receives (at $\theta_{_{YM}} = 0$) identical contributions from instantons and anti-instantons. We will also consider the correlation function of sixteen fermionic superconformal current operators, $\langle \hat{\Lambda}(x_1) \dots \hat{\Lambda}(x_{16}) \rangle$, where $\hat{\Lambda}$ is the product of F^- (the self-dual field strength) and λ (the spin-1/2 gaugino in the $\mathbf{4}$ of $SU(4)$) and the correlation function of eight gaugino bilinears, $\langle \mathcal{E}(x_1) \dots \mathcal{E}(x_8) \rangle$. In contrast to the four-point function of $\mathcal{Q}$'s, these correlators receive non vanishing contributions only from instantons and not from anti-instantons (to leading order in the Yang–Mills coupling). The common feature of all the correlation functions under consideration, as well as many others that are related by supersymmetry, is that they provide precisely the sixteen fermionic zero modes that

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are needed to give a nonzero result in the (anti-)instanton background.

As noted in [27] those effects that are seen from the Yang–Mills perspective as instanton effects will be seen from the bulk point of view as effects due to D-instantons. Explicit D-instanton effects can be extracted from certain terms in the IIB effective action that (for fixed radius L) are of order $(\alpha')^3$ relative to the leading Einstein–Hilbert term and have been derived in [28, 29]. One example of such a term has the form (in the string frame) $(\alpha')^{-1}\int d^{10}X\sqrt{G}e^{-\phi/2}f_4(\tau,\overline{\tau})\mathcal{R}^4$ where τ is the complex scalar field, $\tau = C^{(0)} + ie^{-\phi}$, and $\mathcal{R}^4$ denotes a particular contraction of four ten-dimensional Riemann curvature tensors. The function $f_4(\tau,\overline{\tau})$ is a nonholomorphic Eisenstein series that may be expanded for small string coupling ($\tau_2 = e^{-\phi} \to \infty$) in an infinite series of D-instanton terms, in addition to a tree-level and one-loop contribution [28]. This D-instanton contribution will be reviewed in section 4 as will the analogous contribution to the sixteen-dilatino term, $(\alpha')^{-1}\int d^{10}X\sqrt{G}e^{-\phi/2}f_{16}(\tau,\overline{\tau})\Lambda^{16}$ (where f_{16} transforms with specific holomorphic and antiholomorphic weights), which was obtained in [30]. The many other related terms of the same dimension in the IIB effective action that are of the same order in α' can be obtained in similar fashion.

The leading effects of a single D-instanton that are extracted in this manner will be compared in section 5 with the corresponding $\mathcal{N} = 4$ Yang–Mills supercurrent correlators that were considered in section 3. The correspondence between these expressions will be demonstrated most completely for the sixteen-fermion amplitude. We will see that the form of the leading charge-one D-instanton contribution to the Λ^{16} amplitude matches with the expression for the sixteen-$\hat{\Lambda}$ correlator in $\mathcal{N} = 4$ $SU(2)$ Yang–Mills theory calculated in section 3. Given this correspondence, supersymmetry guarantees that all the related correlators must also agree with their AdS counterparts. Although we have not completed a detailed comparison for all such correlation functions we will indicate in outline how this correspondence should work in the cases described in section 3.

In order to further clarify the correspondence between Yang–Mills instantons and D-instantons, in section 6 we will present the classical D-instanton solution of the type IIB supergravity equations in the $AdS_5 \times S^5$ background. The D-instanton is a solution to the ten-dimensional Euclidean field equations that breaks half the supersymmetries and in which the complex scalar IIB field, τ, has non-trivial spatial dependence. The solution is particularly simple to establish by using the fact that $AdS_5 \times S^5$ is conformally flat, which relates the solution to the flat-space solution of [31]. Although the D-instanton solution does not affect the $AdS_5 \times S^5$ geometry in the Einstein frame it generates a wormhole in the string frame which leads to a rather intriguing modification of the geometry in the limit of large instanton number ($K \to \infty$).

The superpartners of the D-instanton in the gravity supermultiplet are obtained by acting on this solution with the broken supersymmetries and it is striking that the D-instanton induced profiles of these fields match with those of the corresponding

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Yang–Mills supercurrents. This reflects the fact that the moduli space of a $SU(2)$ Yang–Mills instanton is AdS_5 (related comments appear in [32]). We will also verify in section 6 that the leading D-instanton contribution to the Λ^{16} amplitude can be obtained by semiclassical quantization around the classical D-instanton background.

It is encouraging to find this level of agreement between the instanton terms in the IIB superstring and the Yang–Mills theory. Of course, it is difficult to assess without further calculations whether the agreement is qualitative or more precise. The Yang–Mills calculation is based on perturbation theory and with the gauge group $SU(2)$ while agreement with the α' expansion of the IIB theory is only expected in the limit of large N_c and $g^2_{_{YM}} N_c$. We only consider the leading term in the semiclassical expansion around a single-instanton background while there are known to be an infinite number of perturbative corrections even to the one D-instanton expression. For $N_c > 2$ there are additional issues related to the presence of extra zero modes in the semi-classical approximation to the Yang–Mills theory. We will discuss this and other aspects of the generalization to $SU(N_c)$ in the concluding section.

2. Fields and currents in $\mathcal{N} = 4$ SYM

The field content of the maximally supersymmetric ($\mathcal{N} = 4$) four-dimensional Yang–Mills [1, 6, 33] theory is unique apart from the choice of the gauge group. The theory is classically invariant under superconformal transformations as well as under global $SU(4)$ transformations, which form the R-symmetry group of automorphisms of the $\mathcal{N} = 4$ supersymmetry algebra. The field content consists of six real scalars, four Weyl spinors and one vector which are all in the adjoint representation of the gauge group. More precisely, the scalars $\varphi^{AB} = -\varphi^{BA}$ (with $\overline{\varphi}_{AB} = \frac{1}{2}\varepsilon_{ABCD}\varphi^{CD}$) are in the **6** of the $SU(4)$ R-symmetry group.[1] In the following we will also use the notation $\varphi^i = \frac{1}{2}\bar{t}^i{}_{AB}\varphi^{AB}$ $(i = 1, \ldots, 6)$ with $\varphi^i = \varphi^{i*}$ and $\bar{t}^i{}_{AB}$ are Clebsch–Gordan coefficients that couple two **4**'s to a **6** (these are six-dimensional generalizations of the four-dimensional $\sigma^\mu{}_{\alpha\dot\alpha}$ matrices). The spinors λ^A, $\overline{\lambda}_B$ transform as **4** and **4***, respectively, and the vector A_μ is a singlet of $SU(4)$. The classical Minkowskian action is given in terms of these component fields by

$$
\begin{aligned}
S = \int d^4x \,\mathrm{Tr}\Big\{&(D_\mu\varphi^{AB})(D^\mu\overline{\varphi}_{AB}) - \frac{1}{2}i(\lambda^{\alpha A}\overleftrightarrow{\not{D}}_{\alpha\dot\alpha}\overline{\lambda}^{\dot\alpha}{}_A) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \\
&- g_{_{YM}}\lambda^{\alpha A}[\lambda_\alpha{}^B, \overline{\varphi}_{AB}] - g_{_{YM}}\overline{\lambda}_{\dot\alpha A}[\overline{\lambda}^{\dot\alpha}{}_B, \varphi^{AB}] + 2g^2_{_{YM}}[\varphi^{AB}, \varphi^{CD}][\overline{\varphi}_{AB}, \overline{\varphi}_{CD}]\Big\}, \quad (2.1)
\end{aligned}
$$

where Tr denotes a trace over the $SU(N_c)$ colour indices and $D^{ab}_\mu = \delta^{ab}\partial_\mu + ig_{_{YM}} f^{abc}A_{\mu c}$ is the covariant derivative in the adjoint representation.

Although there is no off-shell formulation of $\mathcal{N} = 4$ Yang–Mills theory the on-shell states may be packaged into a superfield W^{AB} that is a function of sixteen Grassmann

[1] The superscripts $A, B = 1, \ldots, 4$ label **4**'s of $SU(4)$ while subscripts label the **4***'s.

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coordinates, θ^A_α and $\overline{\theta}^{\dot\alpha}_A$, satisfying [34] the reality condition

$$\overline{W}_{AB} = \frac{1}{2}\varepsilon_{ABCD}W^{CD}, \tag{2.2}$$

together with the constraint

$$\mathcal{D}^A_\alpha W^{BC} = \mathcal{D}^{[A}_\alpha W^{BC]}, \tag{2.3}$$

where $\mathcal{D}_\alpha$ is the super–covariant derivative. The latter equation constrains the component fields to satisfy the equations of motion. It follows that the supersymmetry transformations on the component fields with parameters $\eta_\alpha{}^A$ and $\overline{\eta}^{\dot\alpha}{}_A$ are given by

$$\begin{aligned}
\delta\varphi^{AB} &= \frac{1}{2}(\lambda^{\alpha A}\eta_\alpha{}^B - \lambda^{\alpha B}\eta_\alpha{}^A) + \frac{1}{2}\varepsilon^{ABCD}\overline{\eta}_{\dot\alpha C}\overline{\lambda}^{\dot\alpha}_D\,,\\
\delta\lambda_\alpha{}^A &= -\frac{1}{2}F^-_{\mu\nu}\sigma^{\mu\nu}{}_\alpha{}^\beta\eta_\beta{}^B + 4i(\not{D}_{\alpha\dot\alpha}\varphi^{AB})\overline{\eta}^{\dot\alpha}_B - 8g_{_{YM}}[\overline{\varphi}_{BC},\varphi^{CA}]\eta_\alpha{}^B\,,\\
\delta A_\mu &= -i\lambda^{\alpha A}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\eta}^{\dot\alpha}{}_A - i\eta^{\alpha A}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\lambda}^{\dot\alpha}{}_A\,.
\end{aligned} \tag{2.4}$$

The classical action (2.1) is superconformally invariant [33] and this property is believed to be preserved at the quantum level thanks to the exact vanishing of the β-function [6]. The Noether currents associated with the superconformal transformations, together with those corresponding to chiral $SU(4)$ transformations, constitute a supermultiplet whose components are bilinears of $W^i \equiv \frac{1}{2}t^i_{AB}W^{AB}$

$$\mathcal{W}^{ij}_{(2)} = \mathrm{Tr}(W^iW^j - \frac{\delta^{ij}}{6}W_kW^k)\,. \tag{2.5}$$

The components of this current superfield are given for the abelian case by [35],

$$\begin{aligned}
\mathcal{T}^{\mu\nu} &= \frac{1}{2}[\delta^{\mu\nu}(F^-_{\rho\sigma})^2 - 4F^{-\mu}{}_\rho F^{-\nu\rho} + \text{ h.c.}] - \frac{1}{2}\lambda^{\alpha A}\sigma^{(\mu}{}_{\alpha\dot\alpha}\overleftrightarrow{\partial}^{\nu)}\overline{\lambda}^{\dot\alpha}_A +\\
&\quad +\delta^{\mu\nu}(\partial_\rho\overline{\varphi}_{AB})(\partial^\rho\varphi^{AB}) - 2(\partial^\mu\overline{\varphi}_{AB})(\partial^\nu\varphi^{AB}) -\\
&\quad -\frac{1}{3}(\delta^{\mu\nu}\Box - \partial^\mu\partial^\nu)(\overline{\varphi}_{AB}\varphi^{AB})\,,\\
\Sigma^\mu{}_{\alpha A} &= -\sigma^{\kappa\nu}\,, F^-_{\kappa\nu}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\lambda}^{\dot\alpha}{}_A + 2i\overline{\varphi}_{AB}\overleftrightarrow{\partial}^\mu\lambda_\alpha{}^B + \frac{4}{3}i\sigma^{\mu\nu}{}_\alpha{}^\beta\partial_\nu(\overline{\varphi}_{AB}\lambda_\beta{}^B)\,,\\
\mathcal{J}^\mu{}_A{}^B &= \overline{\varphi}_{AC}\overleftrightarrow{\partial}^\mu\varphi^{CB} + \overline{\lambda}_{\dot\alpha A}\overline{\sigma}^{\mu\dot\alpha\alpha}\lambda_\alpha{}^B - \frac{1}{4}\delta_A{}^B\lambda^{\alpha C}\sigma^\mu{}_{\alpha\dot\alpha}\overline{\lambda}^{\dot\alpha}{}_C\,,\\
\mathcal{C} &= (F^-{}_{\mu\nu})^2\,,\\
\hat\Lambda^A_\alpha &= -\sigma^{\mu\nu}{}_\alpha{}^\beta F^-{}_{\mu\nu}\lambda_\beta{}^A\,,\\
\mathcal{E}^{AB} &= \lambda^{\alpha A}\lambda_\alpha{}^B\,,\\
\mathcal{B}_{\mu\nu}{}^{AB} &= \lambda^{\alpha A}\sigma_{\mu\nu\alpha}{}^\beta\lambda_\beta{}^B + 2i\varphi^{AB}F^-{}_{\mu\nu}\,,\\
\hat\chi^C_{\alpha\,AB} &= \frac{1}{2}\varepsilon_{ABDE}(\varphi^{DE}\lambda_\alpha{}^C + \varphi^{CE}\lambda_\alpha{}^D)\,,\\
\mathcal{Q}^{AB}{}_{CD} &= \varphi^{AB}\overline{\varphi}_{CD} - \frac{1}{12}\delta^A{}_{[C}\delta^B{}_{D]}\varphi^{EF}\overline{\varphi}_{EF}\,.
\end{aligned} \tag{2.6}$$

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In the non-abelian case there is a trace over the $SU(N_c)$ colour indices and there are additional terms that enter, for example, into the covariantization of all the derivatives. In (2.6), $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor, $\Sigma^{\mu}{}_{\alpha A}$ are the supersymmetry currents and $\mathcal{J}^{\mu}{}_{A}{}^{B}$ the $SU(4)$ R-symmetry currents. The remaining components are obtained by supersymmetry using the equations of motion. They consist of three scalar components ($\mathcal{C}$, $\mathcal{E}^{(AB)}$, $\mathcal{Q}^{ij}$), two fermionic spin-1/2 components ($\hat{\chi}^{C}_{AB}$ and $\hat{\Lambda}^{A}$) and one antisymmetric tensor ($\mathcal{B}_{\mu\nu}{}^{[AB]}$).

It is often useful to decompose the $\mathcal{N}=4$ multiplet in terms of either $\mathcal{N}=1$ or $\mathcal{N}=2$ multiplets. In the $\mathcal{N}=1$ description a $SU(3)\times U(1)$ subgroup of the original $SU(4)$ is manifest and the representations decompose according to $\mathbf{6}\to\mathbf{3}+\mathbf{3}^{*}$, $\mathbf{4}\to\mathbf{3}+\mathbf{1}$. Thus, the $\mathcal{N}=1$ decomposition of the $\mathcal{N}=4$ field supermultiplet consists of one $\mathcal{N}=1$ vector supermultiplet, V, and three $\mathcal{N}=1$ chiral supermultiplets, Φ^{I} ($I=1,2,3$). The vector, A_μ, and the Weyl spinor, λ_α, in V are $SU(3)$ singlets whereas the complex scalars ϕ^{I} and the Weyl spinors ψ^{I} belong to chiral multiplets in the $\mathbf{3}$. The Yukawa couplings and scalar self-interactions are generated by the manifestly $SU(3)$ invariant superpotential

$$\mathrm{W}=\frac{1}{3!}g_{_{YM}}\varepsilon_{_{IJK}}f_{abc}\Phi^{aI}\Phi^{bJ}\Phi^{cK}\,. \qquad (2.7)$$

In the $\mathcal{N}=2$ description the manifest global symmetry is $SU(2)_{\mathcal{V}}\times SU(2)_{\mathcal{H}}\times U(1)$ and the $\mathcal{N}=4$ field supermultiplet decomposes into a $\mathcal{N}=2$ vector multiplet $\mathcal{V}$ and a hypermultiplet, $\mathcal{H}$. We will denote the representations of $SU(2)_{\mathcal{V}}\times SU(2)_{\mathcal{H}}\times U(1)$ by $(\mathbf{r}_{\mathcal{V}},\,\mathbf{r}_{\mathcal{H}})_q$ with the subscript q referring to the $U(1)$ charge and will make use of the following notation for the component fields,

$$\begin{array}{lcl}\mathcal{V} & \to & \lambda^{u}\in(\mathbf{2},\mathbf{1})_{+1},\quad \varphi\in(\mathbf{1},\mathbf{1})_{+2},\quad A_\mu\in(\mathbf{1},\mathbf{1})_{0}\\ \mathcal{H} & \to & \psi^{\dot u}\in(\mathbf{1},\mathbf{2})_{-1},\quad q_{u\dot u}\in(\mathbf{2},\mathbf{2})_{0}\,,\end{array} \qquad (2.8)$$

where $u,\dot u=1,2$. The scalar component $\mathcal{Q}^{AB}{}_{CD}$ of the $\mathcal{N}=4$ current decomposes in the following way in terms of $\mathcal{N}=2$ fields,

$$\begin{array}{lll}\mathcal{Q}^{S0}=q^{S}\varphi & \mathcal{Q}^{S0\dagger}=q^{S}\overline{\varphi} & \mathcal{Q}^{ST}=q^{S}q^{T}-\text{ trace}\\ \mathcal{Q}_{(+)}=\varphi^{2} & \mathcal{Q}_{(-)}=\overline{\varphi}^{2} & \mathcal{Q}_{(0)}=\overline{\varphi}\varphi-\text{ trace}\,,\end{array} \qquad (2.9)$$

where $q_{u\dot u}=q_{_S}\sigma^{^S}{}_{u\dot u}$ and $\sigma^{^S}=(\mathbb{I},\vec\sigma)$, with $\vec\sigma$ the standard Pauli matrices (and $S,T=1,\ldots,4$ are $SU(2)_{\mathcal{V}}\times SU(2)_{\mathcal{H}}$ vector indices).

In calculating the correlation functions in the next section it will be important to understand the systematics of the fermion zero modes associated with the Yang–Mills instanton. As we will see these can either be determined directly from the supersymmetry transformations (2.4) or by making use of the explicit Yukawa couplings which

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are given in $\mathcal{N}=2$ notation by

$$\begin{aligned}\mathcal{L}_Y^{(\mathcal{N}=2)} = \frac{\sqrt{2}}{2} g_{_{YM}} f_{abc} \quad & \Big\{ q^a_{_S} \left(\lambda^{b\alpha u} \sigma^{S}{}_{u\dot{u}} \psi_\alpha{}^{c\dot{u}} + \overline{\psi}^b_{\dot{\alpha}\dot{u}} \overline{\sigma}^{S\,\dot{u}u} \overline{\lambda}^{c\dot{\alpha}}{}_u \right) + \\ & + \varphi^a \left(\varepsilon^{uv} \overline{\lambda}^b_{\dot{\alpha}u} \overline{\lambda}^{c\dot{\alpha}}{}_v + \varepsilon_{\dot{u}\dot{v}} \psi^{b\alpha\dot{u}} \psi_\alpha{}^{c\dot{v}} \right) + \\ & + \overline{\varphi}^a \left(\varepsilon_{uv} \lambda^{b\alpha u} \lambda_\alpha{}^{cv} + \varepsilon^{\dot{u}\dot{v}} \overline{\psi}^b_{\dot{\alpha}\dot{u}} \overline{\psi}^{c\dot{\alpha}}{}_{\dot{v}} \right) \Big\} \,, \end{aligned} \tag{2.10}$$

where $u, v = 1, 2$ are $SU(2)$ indices for the defining representation.

3. Instanton contributions to Yang–Mills correlation functions

We will now consider a selection of current correlation functions in the $\mathcal{N}=4$ Yang–Mills theory which receive contributions in the one-instanton semi-classical approximation. Here and in the following we will restrict to the case of a $SU(2)$ gauge group although we will make general comments about the $SU(N_c)$ case in the concluding section (section 7). The special feature shared by the particular observables that we will consider is that they receive a non vanishing contribution from the $2N_c \cdot \mathcal{N} = 16$ gaugino zero modes that arise in the instanton background.

We will begin by studying four-point correlation functions which are expected to correspond (according to [8]) to non-perturbative IIB interactions that do not violate the ten-dimensional $U_B(1)$ symmetry of the two-derivative part of the effective action. We will then consider examples of correlation functions associated with products of composite operators with fields at the same point and no derivatives (contact terms). These should correspond to supergravity amplitudes that violate the ten-dimensional $U_B(1)$ symmetry but are consistent with $SL(2,\mathbb{Z})$ invariance.

3.1. Correlations of four Yang–Mills supercurrents

An obvious example of a correlation function of four superconformal currents is the correlation function of four stress tensors. However, due to its complicated tensorial structure even the free-field expression for this correlator is awkward to express compactly and it is much simpler to consider correlations of four gauge-invariant composite scalar operators, such as the non-singlet scalars,

$$\mathcal{Q}^{ij} = \varphi^i \varphi^j - \delta^{ij} \varphi_k \varphi^k / 6 \tag{3.1}$$

$(i, j, k = 1, 2, \ldots, 6)$, where $\mathcal{Q}^{ij}$ is the lowest component of the composite twisted chiral current superfield [21] $\mathcal{W}^{ij}_{(2)}$ of equation (2.5). After calculating correlation functions of these currents one can derive those of any other currents in the $\mathcal{N}=4$ supercurrent multiplet by making use of the superconformal symmetry [21]. A way to explicitly do it may be to resort to analytic superspace, recalling that

$$\mathcal{Q}^{ij}(x) \sim \mathcal{W}^{ij}_{(2)}(\Upsilon)\Big|_{\theta=\bar{\theta}=0} \,, \tag{3.2}$$

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where Υ are the supercoordinates of analytic superspace [36]. Therefore, by computing correlators of $\mathcal{Q}^{ij}(x)$ and substituting $x \to \Upsilon$ any of the other correlation functions can in principle be obtained by expanding in the fermionic as well as in the auxiliary bosonic coordinates of analytic superspace.

We will therefore begin by considering

$$\langle \mathcal{Q}^{i_1j_1}(x_1)\mathcal{Q}^{i_2j_2}(x_2)\mathcal{Q}^{i_3j_3}(x_3)\mathcal{Q}^{i_4j_4}(x_4)\rangle \,. \tag{3.3}$$

The value of this correlation function in the free field theory is determined from the expression for the free two-point scalar Green function which is

$$\langle \varphi^{ia}(x)\varphi^{jb}(y)\rangle_{\rm free} = \frac{1}{(2\pi)^2}\frac{\delta^{ij}\delta^{ab}}{(x-y)^2}\,. \tag{3.4}$$

Therefore, the free-field expression for the correlation function that follows by Wick contractions is

$$\begin{aligned}&\langle \mathcal{Q}^{i_1j_1}(x_1)\mathcal{Q}^{i_2j_2}(x_2)\mathcal{Q}^{i_3j_3}(x_3)\mathcal{Q}^{i_4j_4}(x_4)\rangle_{\rm free} = \qquad (3.5)\\ &= \frac{1}{(4\pi^2)^4}\left[{N_c}^2\frac{\delta^{i_1i_3}\delta^{j_1j_3}\delta^{i_2i_4}\delta^{j_2j_4}}{x_{13}^4x_{24}^4} + N_c\frac{\delta^{j_4i_1}\delta^{j_1i_3}\delta^{j_3i_2}\delta^{j_2i_4}}{x_{41}^2x_{13}^2x_{32}^2x_{24}^2} + \text{ permutations}\right],\end{aligned}$$

where,

$$x_{ij} = x_i - x_j\,. \tag{3.6}$$

The first term in this expression is simply the product of two two-point functions and is known to be exact. The second term, which is the free-field contribution to the connected four-point function, certainly gets corrections from interactions. These come from planar Feynman diagrams in the 't Hooft limit, in which $N_c \to \infty$ and $\hat{g}^2 = g_{_{YM}}^2 N_c$ is held fixed [17], and are of order N_c and arbitrary order in $\hat{g}^2$. Since α'/L^2 is assumed to be small the $AdS_5 \times S^5$/Yang–Mills correspondence requires $\hat{g}^2$ be large (see equation (1.1)) and such interaction contributions are not under control from the point of view of the perturbative Yang–Mills theory.

It is easy to check that in the one-instanton background the correlation function (3.3) soaks up the sixteen gaugino zero-modes as follows from the form of the supersymmetry transformations in (2.4). The standard instanton solution gives for the self-dual non-abelian field strength the expression

$$F^{a-}_{(0)\mu\nu} = -\frac{4}{g_{_{YM}}}\frac{\eta^a_{\mu\nu}\rho_0^2}{\left(\rho_0^2+(x-x_0)^2\right)^2}\,, \tag{3.7}$$

where $\eta^a_{\mu\nu}$ is the 't Hooft symbol [37] and the overall power of $g_{_{YM}}$ is consistent with the normalization of the action (2.1). The expression (3.7) is annihilated by the conserved supersymmetry transformations (those associated with the parameter $\overline{\eta}^{\dot\alpha}{}_A$ in (2.4)),

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while the transformations corresponding to supersymmetries associated with $\eta_\alpha{}^A$ are broken and generate eight of the sixteen fermionic zero modes.

In order to evaluate this instanton contribution in the most convenient fashion it will be convenient to use the $\mathcal{N}=2$ supersymmetric description in which as we explained before φ^i decomposes into the complex $\mathcal{N}=2$ singlet φ with $U(1)$ charge $+2$ and a neutral $\mathcal{N}=2$ quaternion $q_{\scriptscriptstyle T}$ in the $(\mathbf{2},\mathbf{2})_0$ representation which resides in the $\mathcal{N}=2$ hypermultiplet. It will then be sufficient to consider the correlation function of two φ^2 and two $\overline{\varphi}^2$ and use $SU(4)$ symmetry to deduce all the other components. From (3.6) the free-field expression for this particular correlator is

$$\langle\varphi^2(x_1)\varphi^2(x_2)\overline{\varphi}^2(x_3)\overline{\varphi}^2(x_4)\rangle_{\rm free} = \frac{1}{(4\pi^2)^4}\left(\frac{4N_c^2}{x_{13}^4x_{24}^4}+\frac{4N_c^2}{x_{14}^4x_{23}^4}+\frac{16N_c}{x_{41}^2x_{13}^2x_{32}^2x_{24}^2}\right), \qquad (3.8)$$

where in the previous notation $\varphi^2=\mathcal{Q}_{(+)}\equiv\mathcal{Q}^{55}-\mathcal{Q}^{66}+2i\mathcal{Q}^{56}$.

In considering the one-instanton contribution to this correlation function one must integrate over the fermionic zero modes that are present in the instanton background. The $\mathcal{N}=2$ formalism is particularly transparent because it follows from the structure of the Yukawa couplings (2.10) that φ only absorbs those zero modes of the $\mathcal{N}=4$ gauginos that belong to the $\mathcal{N}=2$ vector multiplet (λ^u_α) while $\overline{\varphi}$ absorbs the zero modes belonging to the $\mathcal{N}=2$ hypermultiplet $(\psi^{\dot{u}}_\alpha)$. We use the expressions for the zero-modes suggested by the supersymmetry transformations of the instanton

$$\lambda^u_{(0)\alpha}=\frac{1}{2}F^-_{(0)\mu\nu}\sigma^{\mu\nu}_\alpha{}^\beta\frac{1}{\sqrt{\rho_0}}\left(\rho_0\eta^u_\beta+(x-x_0)_\kappa\sigma^\kappa_{\beta\dot{\beta}}\bar{\xi}^{\dot{\beta}u}\right). \qquad (3.9)$$

and similarly for $\psi^{\dot{u}}_{(0)\alpha}$. In equation (3.9) η's and $\bar{\xi}$'s are constant adimensional spinors. In this decomposition η_β and $x_{\beta\dot{\beta}}\bar{\xi}^{\dot{\beta}}$ are the parameters of the broken supersymmetry and special supersymmetry transformations respectively. To recover the natural normalization in which the supersymmetry parameters have dimension $(\text{length})^{1/2}$ we have introduced the appropriate factors of ρ_0. For future use notice that one can assemble the fermionic collective coordinates into the spinors $\zeta^u_\pm(\rho_0,x)$ (where $\pm$ refers to the $U(1)$ R-symmetry charge) to parametrize the fermionic zero modes,

$$\zeta^u_{\pm\alpha}(\rho_0,x-x_0)=\frac{1}{\sqrt{\rho_0}}\left(\rho_0\eta^u_{\pm\alpha}+\sigma^\mu{}_{\alpha\dot{\alpha}}(x_\mu-x_{0\,\mu})\bar{\xi}^{\dot{\alpha}\,u}_\pm\right). \qquad (3.10)$$

It will prove relevant in the AdS/Yang–Mills correspondence that the spinor ζ can be rewritten as a chiral projection of a five dimensional spinor,

$$\zeta(\rho_0,x-x_0)=\left(\frac{1-\gamma^5}{2}\right)\frac{z^M\gamma_M\zeta^{(0)}}{\sqrt{\rho_0}}, \qquad (3.11)$$

where

$$\zeta^{(0)}=\begin{pmatrix}\eta_\alpha\\ \\ \bar{\xi}^{\dot{\alpha}}\end{pmatrix}, \qquad z^M=(x^\mu-x^\mu_0,\rho_0), \qquad (3.12)$$

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and $\gamma^M = (\gamma^\mu, \gamma^5)$ can be interpreted as the five dimensional Dirac matrices in the AdS_5 space (and we have suppressed the $SU(2)_{\mathcal{V}} \times SU(2)_{\mathcal{H}} \times U(1)$ R-symmetry indices u and $\dot{u}$). In fact, as we will see later, these spinors turn out to be chiral projections of the Killing spinors of AdS_5 which is crucial for the correspondence with D-instanton calculations.

With the above normalization conventions, the jacobian of the fermionic part of the instanton measure is (in the $N_c = 2$ case at hand)

$$J_F = \left(\frac{g^2_{YM}}{2^5 \sqrt{2} \pi^2 \rho_0} \right)^8 . \tag{3.13}$$

One can also compute the jacobian for the transformation from bosonic zero-modes to the collective coordinates that leads to the familiar bosonic measure for $SU(N_c)$ instantons [38]

$$d\mu_{B(K=1)} = \frac{4}{\pi^2} \frac{(2\pi)^{4N_c}}{(N_c-1)!(N_c-2)!} \left(\frac{\rho_0}{g_{YM}} \right)^{4N_c} \frac{d\rho_0 d^4 x_0}{\rho_0^5} , \tag{3.14}$$

where we will only make use of the case $N_c = 2$.[2]

Using the preceding equations one finds that in the semi-classical approximation [37, 23] the one-instanton contribution to the four bosonic current Green function is[3]

$$G_{\mathcal{Q}^4}(x_p) = \langle g^2_{YM} \varphi^2(x_1) g^2_{YM} \varphi^2(x_2) g^2_{YM} \overline{\varphi}^2(x_3) g^2_{YM} \overline{\varphi}^2(x_4) \rangle_{K=1} = \frac{g^8_{YM}}{2^{34} \pi^{10}} e^{-\frac{8\pi^2}{g^2_{YM}} + i\theta_{YM}}$$
$$\int \frac{d\rho_0 d^4 x_0}{\rho_0^5} d^4\eta_+ d^4\overline{\xi}_+ d^4\eta_- d^4\overline{\xi}_- \; \varphi^2_{(0)}(x_1) \varphi^2_{(0)}(x_2) \overline{\varphi}^2_{(0)}(x_3) \overline{\varphi}^2_{(0)}(x_4) , \tag{3.15}$$

where the subscript $K = 1$ denotes the winding number of the background and $(+)$ (or $(-)$) refers to the $U(1)$ charges of the gauginos in the vector (or hyper) multiplet (see below). The Green function $G_{\mathcal{Q}^4}$ receives a contribution also from the $K = -1$ sector that is the complex conjugate of (3.15).

In (3.15) the fields φ and $\overline{\varphi}$ have been replaced by the expressions

$$\varphi(x) \to \varphi_{(0)}(x) = \frac{1}{2\sqrt{2}} \varepsilon_{uv} \zeta^u_+ \sigma^{\mu\nu} \zeta^v_+ F^-_{(0)\mu\nu}$$
$$\overline{\varphi}(x) \to \overline{\varphi}_{(0)}(x) = \frac{1}{2\sqrt{2}} \varepsilon_{\dot{u}\dot{v}} \zeta^{\dot{u}}_- \sigma^{\mu\nu} \zeta^{\dot{v}}_- F^-_{(0)\mu\nu} , \tag{3.16}$$

[2]The combined (fermionic and bosonic) subtraction point dependence, $\mu^{(4-\mathcal{N})N_c}$, disappears for $\mathcal{N} = 4$ as well as for all other superconformal theories.

[3]Here and in the following computations of correlators of currents bilinear in the fundamental Yang–Mills fields a factor of g^2_{YM} is included for each external insertion. It would be equivalent to rescale the fundamental Yang–Mills fields according to $A_{YM} \to A'_{YM} = g_{YM} A_{YM}$ and compute the Green functions of primed fields. This leads to a common overall dependence on g^2_{YM} for the three correlation functions that we will consider.

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which are the leading nonvanishing terms that result from Wick contractions in which Yukawa couplings are lowered from the exponential of the action until a sufficient number of fermion fields are present to saturate the fermionic integrals. Of course, these expressions can also be obtained directly from the supersymmetry transformations (2.4) by acting twice on $F^{-}_{(0)\mu\nu}$ with the broken supersymmetry generators. After some elementary Fierz transformations on the fermionic collective coordinates the fermionic integrations can be performed in a standard manner and the result is

$$G_{Q^4}(x_p) = \frac{3^4}{4\pi^{10}}\, g^8_{_{YM}}\, e^{-\frac{8\pi^2}{g^2_{YM}}+i\theta_{YM}} \int \frac{d\rho_0 d^4x_0}{\rho_0^5}\, x^4_{12}x^4_{34} \prod_{p=1}^{4}\left(\frac{\rho_0}{\rho_0^2+(x_p-x_0)^2}\right)^4 . \quad (3.17)$$

As mentioned earlier, the fact that the instanton form factor, $\rho_0^4/[(\rho_0^2+(x_p-x_0)^2]^4$, that enters this expression is identical to K_4 in (1.5) will be of significance in the discussion of the AdS/Yang–Mills correspondence in section 5. The integration in (3.17) resembles that of a standard Feynman diagram with momenta replaced by position differences and can be performed by introducing the Feynman parametrization

$$\begin{aligned} G_{Q^4}(x_p) = & \frac{3^4\Gamma(16)}{4\pi^{10}\left(\Gamma(4)\right)^4}\, g^8_{_{YM}}\, e^{-\frac{8\pi^2}{g^2_{YM}}+i\theta_{YM}} \int \prod_p \alpha_p^3 d\alpha_p \delta\left(1-\sum_q \alpha_q\right)\cdot \\ & \cdot \int \frac{d\rho_0 d^4x_0}{\rho_0^5}\frac{x^4_{12}x^4_{34}\,\rho_0^{16}}{(\rho_0^2+x_0^2-2x_0\cdot\sum_p\alpha_p x_p+\sum_p x_p^2)^{16}} . \end{aligned} \quad (3.18)$$

The five-dimensional integral yields

$$\begin{aligned} G_{Q^4}(x_p) = & \frac{3^3\,\Gamma(11)}{2^7\left(\pi^3\Gamma(4)\right)^4}\, g^8_{_{YM}} e^{-\frac{8\pi^2}{g^2_{YM}}+i\theta_{YM}} \\ & \int \prod_p \alpha_p^3 d\alpha_p\, \delta\left(1-\sum\nolimits_q \alpha_q\right) \frac{x^4_{12}x^4_{34}}{(\sum_p \alpha_p\alpha_q x^2_{pq})^8} . \end{aligned} \quad (3.19)$$

This integral can be simplified by observing that it is essentially obtained by acting with derivatives on the box-integral with four massless external particles,

$$G_{Q^4}(x_p) = \frac{3^3\,\Gamma(11)}{2^7\left(\pi^3\Gamma(4)\right)^4}\, g^8_{_{YM}} e^{-\frac{8\pi^2}{g^2_{YM}}+i\theta_{YM}}\, x^4_{12}x^4_{34}\prod_{p<q}\frac{\partial}{\partial x^2_{pq}} B(x_{pq}) , \quad (3.20)$$

where the box integral is

$$B(x_{pq}) = \int \frac{\prod_p d\alpha_p \delta\left(1-\sum_q\alpha_q\right)}{(\sum_p \alpha_p\alpha_q x^2_{pq})^2} . \quad (3.21)$$

The result may be expressed as a combination of dilogarithms[4]

$$\begin{aligned} B(x_{pq}) = & \frac{1}{\sqrt{\Delta(x_{pq})}}\left[-\frac{1}{2}\log\left(\frac{u_+u_-}{(1-u_+)^2(1-u_-)^2}\right)\log\left(\frac{u_+}{u_-}\right) - \right. \\ & \left. - \ \mathrm{Li}_2(1-u_+) + \mathrm{Li}_2(1-u_-) - \ \mathrm{Li}_2\left(1-\frac{1}{u_-}\right) + \ \mathrm{Li}_2\left(1-\frac{1}{u_+}\right)\right] , \end{aligned} \quad (3.22)$$

[4]We thank Lance Dixon for pointing out a sign error in the version of this formula in [39] and suggesting where to find the correct version in [40].

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where

$$\Delta(x_{pq}) = \det_{4\times 4}((x^2_{pq})) = X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX, \tag{3.23}$$

and

$$u_\pm = \frac{Y + X - Z \pm \sqrt{\Delta}}{2Y}, \tag{3.24}$$

with $X = x^2_{12}x^2_{34}$, $Y = x^2_{13}x^2_{24}$ and $Z = x^2_{14}x^2_{23}$.

Notice that up to an overall dimensional factor needed for the correct scaling, $G_{\mathcal{Q}^4}$ turns out to be a function of the two independent superconformally invariant cross ratios X/Z and Y/Z. Although not immediately apparent, the expression $B(x_{pq})$ is symmetric under any permutation of the external legs, as can be seen by making use of the properties of the dilogarithms

$$\begin{aligned} \mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) &= \frac{\pi^2}{6} - \log(z)\log(1-z)\,, \\ \mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) &= -\frac{\pi^2}{6} - [\log(-z)]^2\,, \end{aligned} \tag{3.25}$$

and observing that the relevant permutations correspond to permutations of $X = Yu_+u_-$, Y and $Z = Y(1-u_+)(1-u_-)$, that are generated by the two transformations: a) $u_+ \to 1/u_-$, $u_- \to 1/u_+$, $Y \to Yu_-u_+$, which is equivalent to the exchange of X and Y, leaving Z fixed (*i.e.* to the exchange of x_1 and x_4 or, equivalently, of x_2 and x_3) and b) $u_+ \to 1 - u_-$, $u_- \to 1 - u_+$ at fixed Y, which is equivalent to the exchange of X and Z (or the exchange of x_2 and x_4 or, equivalently, of x_1 and x_3).

Unlike correlation functions of elementary fields that are infra-red problematic and gauge-dependent, the above correlator is well defined at non-coincident points. Up to the derivatives acting on the box integral, the result is exactly the one expected for the correlator of four scalar operators of dimension $\Delta = 2$ each. The detailed expression for this correlator, including the $g^8_{_{YM}}$ factor, will later be related to the single D-instanton contribution to the $\mathcal{R}^4$ term in the type IIB effective action around the AdS background.

3.2. The correlation function of sixteen fermionic currents

As could have been anticipated, it is particularly simple to analyze the contribution of the Yang–Mills instanton to the correlation function of sixteen of the fermionic superconformal current bilinears, $\hat{\Lambda}^A_\alpha = \mathrm{Tr}(\sigma^{\mu\nu}{}_\alpha{}^\beta F^-_{\mu\nu}\lambda_\beta{}^A)$,

$$G_{\hat{\Lambda}^{16}}(x_p) = \langle \prod_{p=1}^{16} g^2_{_{YM}} \hat{\Lambda}^{A_p}_{\alpha_p}(x_p)\rangle_{K=1}\,. \tag{3.26}$$

Since each factor of $\hat{\Lambda}$ in the product provides a single fermion zero mode it is necessary to consider the product of sixteen currents in order to saturate the sixteen

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grassmannian integrals. To leading order in $g_{_{YM}}$, $G_{\hat\Lambda^{16}}$ does not receive contribution from anti-instantons. The leading term in the one-instanton sector is simply obtained by replacing each $F^-_{\mu\nu}$ with the instanton profile $F^-_{(0)\mu\nu}$ (equation (3.7)) and each $\lambda_\alpha{}^A$ with the corresponding zero mode, $\lambda_{(0)\alpha}{}^A$, that can be deduced from the action of the broken supersymmetry in the second line of (2.4),

$$\lambda^A_{(0)\alpha} = \frac{1}{2} F^-_{(0)\mu\nu} \sigma^{\mu\nu}_\alpha{}^\beta \frac{1}{\sqrt{\rho_0}} \left(\rho_0 \eta^A_\beta + (x-x_0)_\mu \sigma^\mu_{\beta\dot\beta} \bar\xi^{\dot\beta A} \right) . \tag{3.27}$$

The resulting correlation function thus has the form

$$G_{\hat\Lambda^{16}}(x_p) = \frac{2^{62} 3^{16}}{\pi^{10}} g^8_{_{YM}} e^{-\frac{8\pi^2}{g^2_{YM}} + i\theta_{YM}} \int \frac{d^4x_0\, d\rho_0}{\rho_0^5} \int d^8\eta d^8\bar\xi \cdot$$
$$\cdot \prod_{p=1}^{16} \left[\frac{\rho_0^4}{[\rho_0^2 + (x_p - x_0)^2]^4} \frac{1}{\sqrt{\rho_0}} \left(\rho_0 \eta^{A_p}_{\alpha_p} + (x_p - x_0)_\mu \sigma^\mu_{\alpha_p \dot\alpha_p} \bar\xi^{\dot\alpha_p A_p} \right) \right] . \tag{3.28}$$

The integration over the fermion zero modes leads to a sixteen-index invariant tensor, t_{16}, of the product of the $SU(4)$ and Lorentz groups. Assembling the 16 fermionic collective coordinate into a sixteen-dimensional spinor t_{16} would simply read $t_{16}^{a_1 a_2 \ldots a_{16}} = \varepsilon^{a_1 a_2 \ldots a_{16}}$, with $a_i = 1, 2 \ldots 16$. Further integration over the instanton moduli space would determine the dependence on the coordinates x_p. However, we will leave the expression in the unintegrated form (3.28) which will be compared later with the corresponding expression obtained in the IIB string theory in $AdS_5 \times S^5$. Again, the resemblance of the instanton form factor to the AdS Green function will be of significance to the discussion in section 5.

3.3. The correlation function of eight gaugino bilinear currents

In similar fashion it is easy to deduce the one-instanton contribution to other related processes, such as the eight-point correlation function,

$$G_{\mathcal{E}^8}(x_p) = \langle g^2_{_{YM}} \mathcal{E}^{A_1 B_1}(x_1) \ldots g^2_{_{YM}} \mathcal{E}^{A_8 B_8}(x_8) \rangle_{K=1} , \tag{3.29}$$

which also saturates the sixteen fermionic zero-modes present in the $SU(2)$ one-instanton background. To leading order in $g_{_{YM}}$, $G_{\mathcal{E}^8}$ does not receive contribution from anti-instantons. The complete non-abelian expression for $\mathcal{E}^{AB}$ is

$$\mathcal{E}^{AB} = \lambda^{\alpha a A} \lambda_{\alpha a}{}^B + g_{_{YM}} f_{abc}\, t^{(AB)+}_{ijk} \phi^{ia} \phi^{jb} \phi^{kc} , \tag{3.30}$$

but at leading order in the gauge coupling constant only the term proportional to the gaugino bilinear is relevant. In the instanton background the gaugino bilinear is

$$\lambda^{\alpha a A}_{(0)} \lambda_{(0)\alpha a}{}^B = \frac{3 \cdot 2^6}{g^2_{_{YM}}} \frac{\rho_0^4}{(\rho_0^2 + (x-x_0)^2)^4} \zeta^{\alpha A} \zeta_\alpha{}^B . \tag{3.31}$$

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It follows that

$$
\begin{aligned}
G_{\mathcal{E}^8}(x_p) = {} & \frac{3^8 2^{14}}{\pi^{10}}\, g_{_{YM}}^8\, e^{-\frac{8\pi^2}{g_{YM}^2}+i\theta_{YM}} \int \frac{d^4x_0\, d\rho_0}{\rho_0^5} \int d^8\eta d^8\bar{\xi} \,\times \\
& \times \prod_{p=1}^{8} \left[\frac{\rho_0^4}{(\rho_0^2+(x_p-x_0)^2)^4} \frac{1}{\sqrt{\rho_0}} \left(\rho_0 \eta^{A_p}_{\alpha_p} + (x_p-x_0)_\mu \sigma^\mu_{\alpha_p\dot\alpha_p} \bar{\xi}^{A_p\dot\alpha_p}\right) \times \right. \\
& \qquad \left. \times \varepsilon^{\alpha_p\beta_p} \frac{1}{\sqrt{\rho_0}} \left(\rho_0 \eta^{B_p}_{\beta_p} + (x_p-x_0)_\nu \sigma^\nu_{\beta_p\dot\beta_p} \bar{\xi}^{B_p\dot\beta_p}\right)\right] . \qquad (3.32)
\end{aligned}
$$

The integration over the fermion zero modes leads to an $SU(4)$ invariant contraction of sixteen-index tensor t_{16} defined after (3.28) and further integration over the instanton moduli space would determine the exact dependence on the coordinates x_p. Again the unintegrated expression (3.32) is sufficient for comparison with the D-instanton contribution to the corresponding $AdS_5 \times S^5$ amplitude.

4. D-instanton effects in $D = 10$ IIB theory

We now turn to consider the single D-instanton contribution to the amplitudes in the IIB superstring theory that are related to the above Yang–Mills correlation functions according to the correspondence suggested in [8].

The two leading terms in the momentum expansion of the IIB effective action that involve only the metric and the complex scalar fields are given (in string frame) by [28],

$$
\begin{aligned}
(\alpha')^{-4} & \int d^{10}X \sqrt{g}\left(e^{-2\phi}R + k(\alpha')^3 f_4(\tau,\bar\tau) e^{-\phi/2}\mathcal{R}^4\right) = \\
& = L^{-8} \int d^{10}X \sqrt{g}\left((4\pi N_c)^2 R + kL^6 (4\pi N_c)^{1/2} f_4(\tau,\bar\tau)\mathcal{R}^4\right) , \qquad (4.1)
\end{aligned}
$$

where the relations (1.1) have been substituted in the second line and $k = 1/(2^{11}\pi^7)$, as follows, for instance, by direct comparison of the result of [28] with equation (15) of [41]. As discussed in outline in [27] the $f_4(\tau,\bar\tau)\mathcal{R}^4$ interaction contains the D-instanton terms that should be compared with the effects of Yang–Mills instantons. The Riemann curvature enters the $\mathcal{R}^4$ factor in a manner that may be most compactly described by writing it as an integral over a sixteen-component Grassmann spinor,

$$
\mathcal{R}^4 \equiv \int d^{16}\Theta (R_{\Theta^4})^4 , \qquad (4.2)
$$

where

$$
R_{\Theta^4} = \overline{\Theta}\Gamma^{\Lambda_1\Lambda_2\Lambda}\Theta\, \overline{\Theta}\Gamma^{\Lambda_3\Lambda_4}{}_{\Lambda}\Theta\, R_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4} , \qquad (4.3)
$$

which only includes the Weyl tensor piece of the Riemann tensor. Here, $\Gamma^{\Lambda_1\Lambda_2\Lambda_3}$ are the totally antisymmetric products of three ten-dimensional Γ-matrices and the Grassmann parameter Θ^a ($a = 1,\ldots,16$) is a chiral spinor of the ten-dimensional theory. This expresses $\mathcal{R}^4$ as an integral over half of the on-shell type IIB superspace, which accounts

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for the fact that this term has exactly determined properties. In [27] it was pointed out that the $\mathcal{R}^4$ term vanishes in the $AdS_5 \times S^5$ background because it involves a fourth power of the (vanishing) Weyl tensor. The first, second and third functional derivatives of $\mathcal{R}^4$ vanish as well, and as a result one finds no corrections to zero, one, two and three-point amplitudes. However, there is a non-zero four-graviton amplitude arising from this term. The boundary values of these gravitons are sources for various bosonic components of the Yang–Mills current supermultiplet. For example, the components of the metric in the AdS_5 directions couple to the stress tensor, $\mathcal{T}^{\mu\nu}$, whereas the traceless components polarized in the S^5 directions couple to massive Kaluza-Klein states. A linear combination of the trace of the metric on S^5 and the fluctuation of the $R \otimes R$ four-form potential couples to the scalar components, $\mathcal{Q}^{ij}$.

It is easy to pick out all of the other terms of the same dimension that are related to $\mathcal{R}^4$ by supersymmetry by, for example, associating the physical fields with the components of an on-shell IIB superfield [42]. Included among these is a sixteen-fermion interaction [30], that in the string frame reads

$$(\alpha')^{-1} \int d^{10}X \sqrt{g}\, e^{-\phi/2} f_{16}(\tau, \overline{\tau}) \Lambda^{16} + \text{c.c.}\,, \tag{4.4}$$

where Λ is a complex chiral $SO(9,1)$ spinor which transforms under the $U_B(1)$ R-symmetry with charge $3/2$ and the interaction is antisymmetric in the sixteen spinor indices. The function f_{16} is related to f_4 by

$$f_{16} = (\tau_2 \mathcal{D})^{12} f_4\,, \tag{4.5}$$

where $\mathcal{D} = (i\partial/\partial\tau - 2q/\tau_2)$ is the covariant derivative that maps a (q,p) modular form into a $(q+2,p)$ form (where the notation (q,p) labels the holomorphic and anti-holomorphic $SL(2,\mathbb{Z})$ weights of the form). Whereas f_4 transforms with modular weight $(0,0)$ the function f_{16} has weight $(12,-12)$ and therefore transforms with a specific phase under $SL(2,\mathbb{Z})$. This is precisely the phase required to compensate for the $U_B(1)$ transformation of the Λ^{16} factor so that the full expression (4.4) is invariant under $SL(2,\mathbb{Z})$.

All of these terms can be expanded in the limit of small coupling (large $e^{-\phi}$) in the form

$$f_n = a_n \zeta(3) e^{-3\phi/2} + b_n e^{\phi/2} + \sum_{K=1}^{\infty} \mathcal{G}_{K,n} e^{\phi/2}\,, \tag{4.6}$$

where the first two terms (with constant coefficients a_n and b_n) have the form of string tree-level and one-loop terms and $\mathcal{G}_{K,n}$ contains the charge-K instanton and anti-instanton terms. The instanton contribution to $\mathcal{G}_{K,n}$ has the asymptotic expansion in powers of e^{ϕ},

$$\mathcal{G}_{K,n} = \mu(K)(Ke^{-\phi})^{n-7/2} e^{-2\pi K(e^{-\phi} + iC^{(0)})} \left(1 + \sum_{k=1}^{\infty} c_{k,n}^{K} (Ke^{-\phi})^{-k}\right)\,, \tag{4.7}$$

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while the anti-instanton contributions will not be considered here. The coefficients $c^K_{k,n}$ are explicitly given in [30] and

$$\mu(K) = \sum_{m|K} \frac{1}{m^2}\,, \tag{4.8}$$

where $m|K$ denotes the sum over all divisors of K. The expression (4.7) has the appropriate factor of $e^{-2K\pi e^{-\phi}}$ to be associated with a charge-K D-instanton effect. The complete expansion in expression (4.7) represents a series of perturbative fluctuations around a D-instanton where the exact coefficients depend on the precise form of the interaction that involves n type IIB fields. However, the leading term in (4.7) can be written as

$$\mathcal{G}_{K,n} \sim \mathcal{Z}_K (K e^{-\phi})^n\,, \tag{4.9}$$

where $\mathcal{Z}_K$ is independent of which particular interaction term is being discussed and is given by

$$\mathcal{Z}_K = \mu(K)(K e^{-\phi})^{-7/2} e^{-2\pi K(e^{-\phi}+iC^{(0)})}\,, \tag{4.10}$$

$\mathcal{Z}$ should be identified with the contribution of a charge-K D-instanton to the measure in string frame which, up to an overall numerical factor c, we write

$$d\mu^{(s)}_K = c(\alpha')^{-1} d^{10}X\, d^{16}\,\Theta\, \mathcal{Z}_K\,. \tag{4.11}$$

In doing this we are being cavalier about the fact that the full series in (4.7) is not convergent (it is actually an asymptotic approximation to a Bessel function). In the end, consistency of the full theory, particularly with modular invariance, should require considering the complete expression for the instanton contribution.

As observed in [27] the charge-K D-instanton action that appears in the exponent in (4.10) coincides with the action of a charge-K Yang–Mills instanton in the boundary theory which indicates a correspondence between these sources of non-perturbative effects. This idea is reinforced by the correspondence between other factors. For example, after substituting $e^{\phi} = g^2_{_{YM}}/4\pi$ and $\alpha' = L^2 N^{-1/2} g^{-1}_{_{YM}}$, the measure (4.11) contains an overall factor of the coupling constant in the form $g^8_{_{YM}}$. Indeed, this is exactly the power expected on the basis of the $AdS_5 \times S^5$/SYM correspondence since, as we have seen, the one-instanton contribution to the Green functions in the $\mathcal{N}=4$ Yang–Mills theory, considered in section 3, also has a factor of $g^8_{_{YM}}$ arising from the combination of the bosonic and fermionic zero modes norms. We will pursue this issue further in the next section by comparing the leading instanton contributions to IIB superstring amplitudes with the corresponding $\mathcal{N}=4$ current correlators considered in section 3, even though a complete matching would be surprising in $N_c = 2$ case under consideration.

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5. AdS/Yang–Mills correspondence

We will now compare the one-instanton contribution to the supersymmetric Yang–Mills correlation functions presented in section 3 with the amplitudes of the IIB superstring theory with appropriate boundary conditions. In the case of the bulk IIB theory the D-instanton effects may be either extracted directly, as in this section, from the exactly known terms in the effective action or deduced, as in the next section, from the integration of the semi-classical fluctuations around the $AdS_5 \times S^5$ D-instanton solution.

The $AdS_5 \times S^5$ IIB background is endowed with a metric which may be defined in terms of ten-dimensional Cartesian coordinates (x^μ, y^i) $(i = 1, \cdots, 6)$ by

$$ds^2 = \frac{L^2}{\rho^2}(dx \cdot dx + dy \cdot dy) = \frac{L^2}{\rho^2}(dx \cdot dx + d\rho^2) + d{\omega_5}^2 \,, \tag{5.1}$$

where $\rho^2 = y^2$ and $d\omega_5^2$ is the spherically-symmetric constant curvature metric on S^5. The $AdS_5 \times S^5$ background is characterized by the non-vanishing fields,

$$F_{MNPQR} = \frac{1}{L}\varepsilon_{MNPQR} \qquad R_{MNPQ} = -\frac{1}{L^2}(g_{MP}g_{NQ} - g_{MQ}g_{NP}) \tag{5.2}$$

$$F_{mnpqr} = \frac{1}{L}\varepsilon_{mnpqr} \qquad R_{mnpq} = +\frac{1}{L^2}(g_{mp}g_{nq} - g_{mq}g_{np}) \tag{5.3}$$

(recall that upper case Latin indices, $M, N, \ldots = 0, 1, 2, 3, 5$, span the AdS_5 coordinates and lower case Latin indices, $m, n, \ldots = 1, 2, 3, 4, 5$ span the S^5 coordinates). The only non-vanishing components of the Ricci tensor are

$$R_{MN} = -\frac{4}{L^2}g_{MN} \qquad R_{mn} = +\frac{4}{L^2}g_{mn} \,. \tag{5.4}$$

Upon contracting (5.4) with the metric tensor it follows that the scalar curvature vanishes.

This background is maximally supersymmetric (just like the Minkowski vacuum) so there are 32 conserved supercharges that transform as a complex chiral spinor of the tangent-space group, $SO(4,1) \times SO(5)$. In the basis where the ten dimensional Γ_Λ matrices are given by $\Gamma_M = \sigma_1 \otimes \gamma_M \otimes \mathbb{I}$ and $\Gamma_m = \sigma_2 \otimes \mathbb{I} \otimes \gamma_m$, the supersymmetries are generated by the Killing spinors that satisfy

$$D_\Lambda \epsilon - \frac{1}{2L}(\sigma_1 \otimes \mathbb{I} \otimes \mathbb{I})\Gamma_\Lambda \epsilon = 0 \,, \tag{5.5}$$

which follows from the requirement that the gravitino supersymmetry transformation should vanish. In this basis the complex chiral supersymmetry parameter reads

$$\epsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \zeta \otimes \kappa \,, \tag{5.6}$$

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where ζ is a complex four-component $SO(4,1)$ spinor and κ a complex four-component $SO(5)$ spinor. The Killing spinor equation (5.5) has components

$$D_M\zeta - \frac{1}{2L}\gamma_M\zeta = 0\,, \tag{5.7}$$

$$D_m\kappa - i\frac{1}{2L}\gamma_m\kappa = 0\,. \tag{5.8}$$

The Killing spinors have a number of interesting properties. In particular, the Killing spinors κ on S^5 may be used to construct the Kaluza-Klein excitations of all the fields in the IIB gauged supergravity starting from the modes of the massless complex singlet dilaton e^{ϕ}. For future purposes, it is convenient to also consider the spinors that satisfy the Killing spinor equation with the opposite relative sign between the two terms in (5.7) and (5.8). We will denote the former by ζ_+ and κ_+ and the latter by ζ_- and κ_-. In the next section, we will also need the Euclidean continuation of the Killing spinors on AdS_5.

The identification of the lowest lying modes that form the supergravity supermultiplet is given in [43] and is rather involved. Indeed, while the massless dilaton is associated with the constant mode on S^5, *i.e.* with the scalar spherical harmonic Y_ℓ with $\ell = 0$, the other scalars in the supermultiplet are associated with excitations on the 5-sphere. In particular the real scalars Q^{ij} with mass $m^2 = -4/L^2$ in the $\mathbf{20}_{\mathbb{R}}$ of the $SO(6)$ isometry group of S^5 result from a combination of the trace of the internal metric and the self-dual $R\otimes R$ five-form field, $F^{(5)}$, with $\ell = 2$ (Q^{ij} are quadrupole moments of S^5). Similarly the complex scalars E^{AB} with mass $m^2 = -3/L^2$ and their conjugates are associated with the pure two-form fluctuations with $\ell = 1$ of the complexified antisymmetric tensor in the internal directions. The **15** massless vectors $V_M^{[ij]}$ that gauge the $SO(6)$ isometry group are in one-to-one correspondence with the Killing vectors of S^5 and result from a linear combination with $\ell = 1$ of the mixed components of the metric and the internal three-form components of the $R\otimes R$ four-form potential, $C^{(4)}$. The **6** complex antisymmetric tensors $B_{MN}^{[AB]}$ with $m^2 = 1/L^2$, that have peculiar first order equations of motion, result from scalar spherical harmonics with $\ell = 1$. The analysis of the fermions is similar. The **4** dilatini Λ^A with mass $m = -3/(2L)$ are proportional to the internal Killing spinors κ_+. The $\mathbf{20}_{\mathbb{C}}$ spinors χ^A_{BC} with mass $m = -1/(2L)$ correspond to internal components of the gravitino with $\ell = 1$. Finally the supergravity multiplet is completed by the massless $\mathbf{4}^*$ gravitinos Ψ_{MA} which are proportional to the internal Killing spinors κ_-.

The above fields are the ones that act as sources for the superconformal currents (2.6). Higher Kaluza–Klein modes have higher values of ℓ. For example, there are scalar modes like Q^{ij} with $\ell > 2$. Each of these can be put in one-to-one correspondence with a gauge singlet composite operator $W_{(\ell)}$ that starts with

$$W^{(i_1 \ldots i_\ell)}\Big|_{\theta=0} = \mathrm{Tr}\,\phi^{(i_1}\ldots\phi^{i_\ell)} - \text{ traces}\,, \tag{5.9}$$

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which has dimension $\Delta = E_0 = \ell$ and belongs to the ℓ-fold symmetric traceless tensor representation of $SO(6)$. The multiplet contains $256\ell^2(\ell^2-1)/12$ states with different E_0 and $SU(4)$ quantum numbers.

It is important to recall that the non-compact $E_{6(+6)}$ symmetry of the ungauged IIB theory that results from a compactification on T^5 is broken by the gauging of the $SO(6)$ subgroup of $E_{6(+6)}$ leaving an unbroken $SL(2,\mathbb{R})$ which is identified with the original global symmetry of the ten-dimensional theory. The classical gauged supergravity lagrangian possesses a local $Sp(8)$ symmetry inherited from the ungauged theory under which the fermions transform as **8** and the scalar vielbein as a **27** (antisymmetric traceless tensor). But minimizing the scalar potential induced by gauging results in a spontaneous breaking of $Sp(8)$ to $U(4)$. The $U(1)$ factor may be thought as a remnant of the local $U_B(1)$ symmetry of the flat ten-dimensional classical supergravity theory, but there is no corresponding symmetry in the $D=4$ Yang–Mills boundary theory. This fits in with the fact that in IIB string theory the $SL(2,\mathbb{R})$ is replaced by $SL(2,\mathbb{Z})$ and the continuous $U_B(1)$ symmetry is not present. For example, the presence of the Λ^{16} interaction in the IIB effective action is consistent with $SL(2,\mathbb{Z})$ but not with the $SL(2,\mathbb{R})$ symmetry of the minimal classical IIB supergravity.

We may now compare the effects of Yang–Mills instantons in supercurrent correlators evaluated in section 3 with those of D-instantons in the $AdS_5\times S^5$ IIB string theory. For illustrative purposes the simplest choice is to compare the D-instanton contribution to the amplitude for sixteen spin-1/2 gluinos, Λ, in the $AdS_5\times S^5$ background with the Yang–Mills instanton contribution to the sixteen-$\hat{\Lambda}$ correlation function. From either perspective the leading instanton contribution arises from the product of sixteen factors each carrying one single fermionic zero mode.

The first method for obtaining this amplitude is to expand the function f_{16} in (4.5) to extract the one-instanton term. In order to compare with the Yang–Mills sixteen-point correlator we need to consider the situation in which all sixteen fermions propagate to well defined configurations on the boundary. The Dirac operator acting on spin-1/2 fields in AdS_5 was given in [44] by

$$\not{D}\Lambda = e_{\hat{L}}{}^{M}\gamma^{\hat{L}}\left(\partial_M + \frac{1}{4}\omega_M^{\hat{M}\hat{N}}\gamma_{\hat{M}\hat{N}}\right)\Lambda = (\rho\gamma^{\hat{5}}\partial_5 + \rho\gamma^{\hat{\mu}}\partial_\mu - 2\gamma^{\hat{5}})\Lambda\,, \tag{5.10}$$

where $e_{\hat{L}}{}^{M}$ is the vielbein, $\omega_M^{\hat{M}\hat{N}}$ the spin connection (hatted indices refer to the tangent space) and $\gamma^{\hat{\mu}}$ are the four-dimensional Dirac matrices. Equation (5.10) leads to the normalized bulk-to-boundary propagator of the fermionic field Λ of mass $m=-3/2L$, associated to the composite operator $\hat{\Lambda}$ of dimension $\Delta=\frac{7}{2}$,

$$K^F_{7/2}(\rho_0,x_0;x) = K_4(\rho_0,x_0;x)\frac{1}{\sqrt{\rho_0}}\left(\rho_0\gamma_{\hat{5}} + (x_0-x)^\mu\gamma_{\hat{\mu}}\right)\,, \tag{5.11}$$

which, suppressing all spinor indices, leads to

$$\Lambda_J(x_0,\rho_0) = \int d^4x K^F_{7/2}(\rho_0,x_0;x)J_\Lambda(x)\,, \tag{5.12}$$

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where $J_\Lambda(x)$ is a left-handed boundary value of Λ and acts as the source for the composite operator $\hat{\Lambda}$ in the boundary $\mathcal{N}=4$ Yang–Mills theory. As a result, the classical action for the operator $(\Lambda)^{16}$ in the $AdS_5\times S^5$ supergravity action is

$$S_\Lambda[J] = e^{-2\pi(\frac{1}{g_s}+iC^{(0)})} g_s^{-12}\, V_{S^5} \int \frac{d^4x_0 d\rho_0}{\rho_0^5}\cdot$$
$$\cdot t_{16}\prod_{p=1}^{16}\left[K_4(\rho_0,x_0;x_p)\frac{1}{\sqrt{\rho_0}}\left(\rho_0\gamma^{\hat{5}}+(x_0-x_p)^\mu\gamma_{\hat{\mu}}\right) J_\Lambda(x_p)\right], \qquad (5.13)$$

where we have set $e^\phi = g_s$ and $C^{(0)} = \tilde{C}^{(0)}$ (since the scalar fields are taken to be constant in the $AdS_5\times S^5$ background) and $V_{S^5} = \pi^3$ is the S^5 volume. The 16-index invariant tensor t_{16} is the same as the one defined after (3.28). The overall power of the coupling constant comes from the factor of g_s^4 in the measure (4.11) and the factor of g_s^{-16} from the leading term in (4.7) (and we have dropped an overall numerical constant). Using the dictionary (1.1) and differentiating with respect to the chiral sources this result agrees with the expression (3.28) obtained in the Yang Mills calculation, including the power of $g_{_{YM}}$ (but we have not checked the overall numerical constant). In the next section we will also motivate this expression directly by semi-classical quantization around a D-instanton field configuration that is a Euclidean solution of the IIB supergravity.

The agreement of the Λ^{16} amplitude with the corresponding sixteen-current correlation function of section 3 is sufficient to guarantee that the instanton contributions to all the other Yang–Mills correlation functions that are related to this term by $\mathcal{N}=4$ supersymmetry will also agree with their IIB superstring counterparts. For example, the correlation function that we considered in most detail in section 3 was the one with four superconformal scalar currents $\mathcal{Q}^{ij}$ which are in the $\mathbf{20}_{_\mathbb{R}}$ of $SU(4)$, and have dimension $\Delta = 2$ and AdS mass $m^2 = -4/L^2$. The supergravity field, Q^{ij}, that couples to $\mathcal{Q}^{ij}$ is a linear combination of the fluctuation of the trace of the metric on S^5, $h_m^{\ m}$, and of the four-form field potential, $C^{(4)}_{MNPQ} = \epsilon_{MNPQR}\nabla^R f$. Therefore, contributions to the correlation of four of these composite scalars in the one D-instanton background should correspond to the leading parts of the $K=1$ terms in the expansion of the $\mathcal{R}^4$ interaction (4.2) as well as terms of the form $\mathcal{R}^2(\nabla F_5)^2$ and $(\nabla F_5)^4$. These last two terms involve F_5, the self-dual field strength of the antisymmetric four-form potential, and are related by supersymmetry to the $\mathcal{R}^4$ term.

It follows from the structure of (3.17) that the Yang–Mills instanton contribution to each factor of $\mathcal{Q}^{ij}$ is of the form $\nabla\nabla K_2(x_0^\mu,\rho_0;x^\mu,0)$, where the two derivatives are not necessarily contracted. But this is the expected form for a propagator from the $AdS_5\times S^5$ bulk to the boundary for a scalar field of dimension 2. Therefore, at least the general form of the expression obtained from the $K=1$ terms in the expansion of $\mathcal{R}^4$ and the related F_5 interactions agrees with the four-$\mathcal{Q}^{ij}$ correlation function in a Yang–Mills instanton background. In order to see this agreement in more detail it would

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first be necessary to determine the precise form of the $(\nabla F_5)^2$ and $(\nabla F_5)^4$ interactions that contain the fluctuations of F_5. The $K=1$ contribution to the amplitude of four fluctuations of the appropriate combination of $h_m{}^m$ and f can then be extracted from these interaction terms. Although we have not so far performed this explicitly, the result is guaranteed to reproduce the $K=1$ expression obtained from the Yang–Mills theory in section 3 since it is related by supersymmetry to the Λ^{16} amplitude.

The analogous comparison of the correlation function of eight $\mathcal{E}^{AB}$'s with the amplitude for eight E^{AB}'s in the IIB superstring theory proceeds in much the same way. The supergravity fields, E^{AB}, that couple to $\mathcal{E}^{AB}$ arise from the internal components of the complex IIB antisymmetric tensor. Supersymmetry relates an H^8 term, where H is the complex IIB three-form field-strength, to the $\mathcal{R}^4$ term in the IIB effective action. One thus expects interactions schematically of the form $(\nabla K_3)^8$, with various contractions of the derivatives. Using the explicit form of the spinor collective coordinates $\zeta^A = (\rho_0\eta^A + (x-x_0)_\mu\sigma^\mu\bar{\xi}^A)/\sqrt{\rho_0}$ one may check that the gaugino bilinears $\mathcal{E}^{AB}$ in the one-instanton background exactly give rise to K_3 and derivatives thereof, as expected for the scalar propagator of a field of dimension $\Delta = 3$ and AdS mass $m^2 = -1/L^2$. Although a precise matching of the resulting amplitudes is still under study, one may appeal to supersymmetry arguments to determine the complete structure of these terms.

Notice that the above considered non-perturbative terms in the IIB effective action when expanded around the $AdS_5 \times S^5$ give rise to both derivative and non-derivative interaction terms. The matching of the non-derivative "mass-related" terms with corresponding terms in the Yang–Mills Green functions is rather trivial but clearly it is only a hint to the conjectured correspondence.

6. The D-instanton solution in $AdS_5 \times S^5$

We will now consider to what extent the information about the charge-one D-instanton term extracted from the $(\alpha')^{-1}$ terms in the IIB effective action can be determined by semi-classical IIB supergravity field theory in a D-instanton background. Recall that in flat ten-dimensional euclidean space the charge-K D-instanton solution is a finite-action euclidean supersymmetric (BPS–saturated) solution in which the metric is trivial ($g_{\mu\nu} = \eta_{\mu\nu}$ in the Einstein frame) but the complex scalar $\tau = C^{(0)} + ie^{-\phi}$ has a nontrivial profile with a singularity at the position of the D-instanton. The (euclidean) $R\otimes R$ scalar is related to the dilaton by the BPS condition $\partial_\Sigma C^{(0)} = \pm i\partial_\Sigma e^{-\phi}$, while the dilaton solution is the harmonic function (correcting an error in [31])

$$e^{\hat{\phi}^{(10)}} = g_s + \frac{3K\alpha'^4}{\pi^4|X-X_0|^8}\,. \tag{6.1}$$

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This is the classical solution of the ten-dimensional Laplace equation, $\partial^2 e^{\phi} = 0$, outside an infinitesimal sphere centered on the point X_0^Λ (where X^Λ is the ten-dimensional coordinate and X_0^Λ is the location of the D-instanton), g_s is the asymptotic value of the string coupling and the normalization of the second term has a quantized value by virtue of a condition analogous to the Dirac–Nepomechie–Teitelboim condition that quantizes the charge of an electrically charged p-brane and of its magnetically charged p'-brane dual [45, 46]. It is notable that the solution in (6.1) is simply the Green function for a scalar field to propagate from X_0 to X subject to the boundary condition that $e^{\phi} = g_s$ at $|X| \to \infty$ or $|X_0| \to \infty$.

We are now interested in solving the equations of motion of the IIB theory in euclidean $AdS_5 \times S^5$. The BPS condition for a D-instanton in this background again requires $\partial_\Sigma e^{-\phi} = \pm i \partial_\Sigma C^{(0)}$ that leads to

$$e^{-\phi} g^{\Lambda\Sigma} \nabla_\Lambda \nabla_\Sigma e^{\phi} = 0 \,, \tag{6.2}$$

and (in the Einstein frame) the Einstein equations are unaltered by the presence of the D-instanton (because the associated Euclidean stress energy tensor vanishes) so that $AdS_5 \times S^5$ remains a solution. Equation (6.2) is identical to the equation for the Green function of a massless scalar propagating between the location of the D-instanton (x_0^μ, y_0^i) and the point (x^μ, y^i), which is the bulk-to-bulk propagator (subject to the boundary condition that it is constant in the limits $\rho \to 0$ and $\rho \to \infty$). This is easy to solve using the conformal flatness of $AdS_5 \times S^5$ which implies that the solution for the dilaton is of the form[5]

$$e^{\hat\phi} = g_s + \frac{\rho_0^4 \rho^4}{L^8} \left(e^{\hat\phi^{(10)}} - g_s \right) , \tag{6.3}$$

where $\rho_0 = |y_0|$ and $e^{\hat\phi^{(10)}}$ is the harmonic function that appeared in the flat ten-dimensional case, (6.1).[6] In evaluating D-instanton dominated amplitudes we will only be interested in the case in which the point (x^μ, y^i) approaches the boundary ($\rho \equiv |y| \to 0$), in which case it is necessary to rescale the dilaton profile (just as it is necessary to rescale the scalar bulk-to-bulk propagator, [9, 10]) so that the combination

$$\rho^{-4} \left(e^{\hat\phi} - g_s \right) = \frac{3K(\alpha')^4}{L^8 \pi^4} \frac{\rho_0^4}{((x - x_0)^2 + \rho_0^2)^4} \,, \tag{6.4}$$

will be of relevance in the $\rho \to 0$ limit.

As mentioned earlier, the correspondence with the Yang–Mills instanton follows from the fact that $\rho_0^4/((x - x_0)^2 + \rho_0^2)^4 = K_4$ is proportional to the instanton number

[5] We are grateful to G.W. Gibbons and M.J. Perry for discussions about the general form of this solution.

[6] This does *not* agree with the expressions recently proposed in [47] or in [48] that appeared while this paper was in preparation.

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density, $(F^-_{(0)})^2$, in the $\mathcal{N}=4$ Yang–Mills theory. Strikingly, the scale size of the Yang–Mills instanton is replaced by the distance ρ_0 of the D-instanton from the boundary. This is another indication of how the geometry of the Yang–Mills theory is encoded in the IIB superstring. Note, in particular, that as the D-instanton approaches the boundary $\rho_0 \to 0$, the expression for $\rho^{-4} e^{\hat\phi}$ reduces to a δ function that corresponds to a zero-size Yang–Mills instanton.

The BPS condition implies that we can write the solution for the $R \otimes R$ scalar as

$$\hat C^{(0)} = \tilde C^{(0)} + i f(x,y)\,, \tag{6.5}$$

where $\tilde C^{(0)}$ is the constant real part of the field (which corresponds to $\theta_{YM}/2\pi$ see equation (1.2)) and

$$f = A - \frac{1}{g_s} + e^{-\hat\phi}\,. \tag{6.6}$$

Since the action is independent of constant shifts of $C^{(0)}$ it does not depend on the arbitrary constant, A. In a manner that follows closely the flat ten-dimensional case considered in the appendix of [28] the action for a single D-instanton of charge K can be written as

$$S_K = -\frac{L^{10}}{(\alpha')^4} \int \frac{d\rho d^4x d^5\omega}{\rho^5} g^{\Lambda\Sigma} \nabla_\Lambda (e^{2\hat\phi} f \partial_\Sigma f)\,, \tag{6.7}$$

which reduces to an integral over the boundaries of $AdS_5 \times S^5$ and the surface of an infinitesimal sphere centered on the D-instanton at $x = x_0$, $y = y_0$. With the choice $A = 0$ in (6.6) the entire D-instanton action comes from the boundary of the infinitesimal sphere. Substituting for f from (6.6) gives

$$S_K = \frac{2\pi|K|}{g_s}, \tag{6.8}$$

which is the same answer as in the flat ten-dimensional case. On the other hand, with the choice $A = 1/g_s$ in (6.6) the expression (6.7) reduces to an integral over the boundary at $\rho = 0$ but the total action remains the same as S_K in (6.8). Remarkably, in this case the boundary integrand is *identical* to the action density of the standard four-dimensional Yang–Mills instanton.

Whereas the $AdS_5 \times S^5$ metric remains unchanged by the presence of the D-instanton in the Einstein frame it is radically altered in the string frame where the instanton is manifested as a space-time wormhole (as in the flat ten-dimensional case [31]). For finite values of K the dilaton becomes large in the Planck-scale neck and the classical solution is not reliable in that region. However, for very large instanton number, the neck region becomes much larger than the Planck scale so, by analogy with the D-brane examples studied in [8], it should be very interesting to study the implications of the modified $AdS_5 \times S^5$ geometry in the large-K limit of the large-N_c theory.

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The D-instanton contribution to the amplitude with sixteen external dilatinos, Λ^A_α, may now be obtained directly by semi-classical quantization around the classical D-instanton solution in $AdS_5 \times S^5$. The leading instanton contribution can be determined by applying supersymmetry transformations to the scalar field which has an instanton profile given by (6.4). Since the D-instanton background breaks half the supersymmetries the relevant transformations are those in which the supersymmetry parameter corresponds to the Killing spinors for the sixteen broken supersymmetries. These Killing spinors have $U_B(1)$ charge 1/2 and are defined by a modified version of (5.5) that includes the non-trivial composite $U_B(1)$ connection, Q_M [49], that is made from the IIB scalar field [31],

$$\mathcal{D}_M \zeta \equiv (D_M - \frac{i}{2} Q_M)\zeta = \frac{1}{2L}\gamma_M \zeta \, . \tag{6.9}$$

Substituting the euclidean D-instanton solution into the expression for the composite connection gives

$$Q_M = \frac{i}{2} e^{-\hat\phi}\partial_M e^{\hat\phi} \, , \tag{6.10}$$

with $\hat\phi$ defined by (6.4). The solution of (6.9) is

$$\zeta_\pm = e^{-\hat\phi/4} \frac{z_M \gamma^{\hat M}}{\sqrt{\rho_0}} \zeta_\pm^{(0)} \, , \tag{6.11}$$

where $\zeta_\pm^{(0)}$ is a constant spinor satisfying $\gamma_5 \zeta_\pm^{(0)} = \pm \zeta_\pm^{(0)}$.

The sixteen broken supersymmetry transformations associated with $\zeta_-^{(0)}$ give rise to the dilatino zero-modes,

$$\Lambda_{(0)} = \delta\Lambda = (\gamma^M \hat P_M)\zeta_- \, , \tag{6.12}$$

where $\hat P_M$ is the expression for $P_M \equiv i\partial_M \tau^*/2\tau_2$ in the D-instanton background [28],

$$\hat P_M = e^{-\hat\phi}\partial_M e^{\hat\phi} \, . \tag{6.13}$$

Using the Killing spinor equation and the D-instanton equation $\mathcal{D}^M \hat P_M = 0$ it is easy to check (recalling that P_M has $U_B(1)$ charge 2) that

$$\gamma^M \mathcal{D}_M \Lambda_{(0)} = -\frac{3}{2L} \, , \Lambda_{(0)}, \tag{6.14}$$

so that $\Lambda_{(0)}$ is a solution of the appropriate massive Dirac equation. We will be interested in amplitudes with external states located on the boundary, in which case we may use the fact that for $\rho \sim 0$,

$$P_M \sim \frac{1}{g_s}\partial_M e^{\hat\phi} \tag{6.15}$$

in (6.12), which leads to

$$\Lambda_{(0)} \sim \frac{4\rho^4}{g_s}(e^{\hat\phi} - g_s)\zeta_- \, . \tag{6.16}$$

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This means that near $\rho = 0$ the dilatino profile in the D-instanton background is proportional to $\rho^4 K_4(x_0, \rho_0; x, 0)$.

As a result the leading contribution to the sixteen-dilatino amplitude again reproduces the corresponding sixteen-current correlator in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. Explicitly, the D-instanton approximation to the amplitude with sixteen external dilatinos, ${\Lambda_\alpha}^A$, at points on the $\rho = 0$ boundary is (up to an overall constant factor)

$$\begin{aligned}\langle \prod_{p=1}^{16} \Lambda^{A_p}_{\alpha_p}(x_p, 0)\rangle_J = g_s^{-12} e^{-2\pi K(\frac{1}{g_s}+iC^{(0)})} V_{S^5} \int \frac{d^4x_0 d\rho_0}{\rho_0^5} \int d^{16}\zeta_-^{(0)} \cdot \\ \cdot \prod_{p=1}^{16} \left[K_4\left(x_0, \rho_0; x_p\right) \frac{1}{\sqrt{\rho_0}} \left(\rho_0 \eta^{A_p}_{\alpha_p} + (x_p - x_0)_\mu \sigma^\mu_{\alpha_p \dot\alpha_p} \bar\xi^{\dot\alpha_p A_p}\right) J_\Lambda(x_p)\right], \qquad (6.17)\end{aligned}$$

where $J_\Lambda(x_p)$ is the wave-function of the dilatino evaluated at the boundary point $(x_p, 0)$ and the Grassmann spinor $\zeta_-^{(0)}$ was defined in terms of η and $\bar\xi$ in (3.12). The power of g_s^{-12} has been inserted in (6.17) from the expression obtained in section 4 although this power should also follow directly by considering the normalization of the bosonic and fermionic zero modes. Up to the overall constant factor, the amplitude (6.17) agrees with (5.13) and therefore with (3.28).

In similar manner the instanton profiles of all the fields in the supergravity multiplet follow by applying the broken supersymmetries to P_M any number of times, just as they do in the flat ten-dimensional case [28]. The single D-instanton contributions to any correlation function can then be determined. This is guaranteed to agree with the corresponding term in the expansion of the effective type IIB action as well as with the corresponding $\mathcal{N} = 4$ Yang–Mills correlation function.

7. Discussion

We have analyzed instanton contributions to correlation functions of superconformal currents in $\mathcal{N} = 4$ Yang–Mills theory in three ways. The first was a direct Yang–Mills instanton calculation at lowest order in perturbation theory around the one-instanton configuration. The superconformal correlation functions that we considered are ones in which the sixteen gaugino zero modes implied by supersymmetry in the instanton background are soaked up by the currents. We considered the particular example of the four-$\mathcal{Q}$ correlation function in most detail, reducing it to an explicit conformal invariant expression involving dilogarithms. We also obtained integral representations for the sixteen-$\hat\Lambda$ and eight-$\mathcal{E}$ correlation functions. These correlation functions and many others are related to each other by the superconformal symmetry.

The Yang–Mills calculations were then compared with single D-instanton contributions to corresponding amplitudes in the $AdS_5 \times S^5$ compactification of the IIB

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superstring. These contributions were isolated from the exactly known form of the appropriate interaction terms in the IIB effective action. The sixteen-Λ amplitude was considered in detail since it is the simplest to compare with the corresponding Yang–Mills expression.

Finally, we derived the classical D-instanton solution in the $AdS_5 \times S^5$ background. Quite strikingly we saw that the D-instanton action could be expressed as a surface integral in two equivalent ways. In one of these the action is localized at the position of the D-instanton in the ten-dimensional euclidean space, (x_0, y_0), while in the other the action is given by an integral over the AdS boundary at $\rho = 0$. In the latter case the integrand is precisely the same as the action density of a Yang–Mills instanton in four euclidean dimensions. We then saw that the sixteen-Λ amplitude obtained by semi-classical quantization agreed with the exact expression (although we have not determined the precise constant factor in the measure).

The fact that the Yang–Mills instanton and IIB D-instanton effects appear to match so closely should be interpreted as general support for the conjectured correspondence of [8]. Notably we have seen that the Yang–Mills instanton scale size has a natural interpretation as the position of the D-instanton in extra dimensions transverse to the four-dimensional boundary space-time. This reflects the fact that the single-instanton moduli spaces in both cases contain an AdS_5 factor. The additional dimensions, which are not at all apparent in standard $\mathcal{N} = 4$ Yang–Mills perturbation theory, are very natural when the $AdS_5 \times S^5$ is interpreted as the near-horizon geometry of N_c D3-branes as in [8]. The points in the space transverse to the boundary, with coordinates y^i, are then identified with the expectation values of the six massless scalar fields in a separated test D3-brane and $\rho^2 = |y|^2$. The five angular coordinates of S^5, ω, are the additional massless scalar fields that complete the moduli space to $AdS_5 \times S^5$.

However, the instanton effects were obtained at leading order in $g_s = 4\pi g^2_{_{YM}}$ and next-to-leading order in $(g^2_{_{YM}} N_c)^{-1/2} = \alpha' L^{-2}$ and should therefore only match precisely for gauge group $SU(N_c)$ in the limit of large-N_c. Since we have only considered the case $N_c = 2$ there are many obvious questions and it is likely that the agreement we have found can only be made precise by extending the calculations in several directions. Firstly, the exact expressions for the IIB interactions include an infinite number of fluctuations around the D-instanton and it would be an interesting project in its own right to study the corresponding series of fluctuations around the Yang–Mills instanton. Secondly, it would be interesting to consider larger $SU(N_c)$ gauge groups. This requires a careful discussion of zero modes. For example, consider the single instanton contribution when $N_c > 2$. The number of bosonic collective coordinates is known to be $4N_c$, but only five of these correspond to translations and dilatation whereas the others are associated with global $SU(2)$ or coset rotations of the instanton inside $SU(N_c)$. For correlators of gauge-invariant operators the latter integrations are trivial and one ends up with a five-dimensional integral over the moduli of the $SU(2)$ one-instanton

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configurations and this coincides with AdS_5. Since there are $8N_c$ fermionic zero modes in the instanton background it might superficially appear that the correlation functions considered earlier would vanish. But this neglects the fact that all but 16 of these are effectively eliminated by the effects of the Yukawa, gauge and scalar self couplings of the fields. We have so far used the fact that the fundamental fields, A, λ, ϕ and $\overline{\lambda}$ absorb, respectively, $0, 1, 2, 3$ fermionic zero modes. However, when $N_c > 2$ these fields have more fermionic zero modes. Schematically, we may write

$$\begin{aligned} DF &= g_{_{YM}} \left(\overline{\lambda}\lambda + \phi D\phi\right) , \\ D\lambda &= g_{_{YM}}\phi\overline{\lambda} , \\ D^2\phi &= g_{_{YM}} \left(\lambda\lambda + \overline{\lambda\lambda}\right) + g_{_{YM}}^2\phi^3 , \\ D\overline{\lambda} &= g_{_{YM}}\phi\lambda . \end{aligned} \tag{7.1}$$

Once the lowest order solution for A, λ, ϕ and $\overline{\lambda}$ is plugged into (7.1) one finds new contributions with *four* more zero-modes must be added to each field. The process ends after exactly $2N_c$ cycles due to the anticommuting nature of the fermionic collective coordinates. Thus, it should be straightforward to generalize the $K = 1$ discussion to gauge groups with $N_c > 2$. It may considerably simplify the analysis to work in a manifestly supersymmetric context, such as that described in [51, 21], which allows for a fully non-linear analytic superspace extension of the Yang–Mills instanton configuration.

The contributions of multiply-charged ($K > 1$) instanton configurations are subtle for $N_c > 2$. The instanton moduli space, including the three global $SU(2)$ rotations, is a hyperkähler quotient defined by the ADHM equations. The process of absorbing the fermionic zero-modes appears to end up with an integration over a much larger space than AdS_5. However, one may speculate that in the large N_c limit the K-instanton measure is concentrated around charge-m $SU(2)$ instantons (where m is a divisor of K) that are embedded in K/m commuting $SU(2)$ subgroups of $SU(N_c)$. This would reduce the domain of integration to symmetric products of K/m AdS_5 factors. These spaces have orbifold singularities at which the number of fermionic zero modes may be reduced so that the special correlators we have considered may get non-zero support only from these regions since they correspond to local effective interactions. Clearly, it would be very interesting to deduce the precise charge-K D-instanton weight, $\mu(K)$ (given by (4.8)), that enters into the IIB interactions by considering such Yang–Mills instanton embeddings in $SU(N_c)$ for large N_c. Presumably, this would shed light on how the moduli space of multiply-charged D-instantons on $AdS_5 \times S^5$ can be represented by the Yang–Mills instantons beyond the most obvious AdS_5 factors.

One approach to determining this measure might be to view the instanton-dominated supercurrent correlators as a topological subset of the $\mathcal{N} = 4$ Yang–Mills theory. The modular-covariant non-holomorphic functions of the complexified couplings of the D-instanton description appear to be analogous to expressions found in the computation

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of the Witten index for topologically twisted $\mathcal{N} = 4$ theories on curved manifolds [25]. Other superconformally invariant gauge theories with fewer supercharges [53] may similarly give rise to interesting topological subsectors in which the correlation functions are dominated by instanton configurations analogous to those that enter the computations performed in this paper.

In summary, the nonperturbative instanton effects that we have considered in this paper seem to be very naturally adapted to the possibility that the dynamics of $\mathcal{N} = 4$ Yang–Mills theory can be considered to be governed by a theory living on the boundary of $AdS_5 \times S^5$ [8], although this correspondence is very obscure in perturbation theory. Even for the gauge group $SU(2)$ the Yang–Mills instanton scale is naturally identified with the distance of the D-instanton from the boundary. This appears to be yet more evidence, albeit qualitative since we have only studied the $N_c = 2$ case, in favour of the Maldacena conjecture.

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